between 10 and 20 and with intense wall cooling. The correlation line obtained is of the form

$$St_{max} = 0.2(Re_{x_p})^{-0.5}$$

The major source of difficulty when attempting to apply this correlation in a practical case lies in the evaluation of x_p. Bushnell and Weinstein make the assumption that the separation geometry is known and propose that if the peak heating occurs near the peak pressure, then from Fig. 1 the following relation can be applied:

$$x_p = [\delta_s/\sin(\theta_f - \theta_s)]$$

Even with schlieren photographs of the flow, small errors in $(\theta_f - \theta_s)$ are bound to occur and when $\sin(\theta_f - \theta_s)$ is a small quantity, large errors can arise in the estimation of x_p. Therefore, in general engineering applications the correlation is difficult to use.

This Note describes some preliminary results of a theoretical investigation which basically support Bushnell and Weinstein's ideas but which indicate a more direct method of predicting the peak heat-transfer coefficients. The theory used is Gautier and Ginoux[2] improved version of Klineberg's[3] integral method for viscous-inviscid interactions in moderate supersonic flows.

3. Results and Discussion

The calculations were carried out for a freestream Mach number of 6.0 and a Reynolds number range of 5×10^6 to 3×10^7/m. The flap angle θ_f was 7.5° and the hinge line was 40 mm from the leading edge. Two values of total enthalpy function s_w were used, −0.4 and −0.8, where $s_w = (T_w/T_{0_\infty}) - 1$.

Peak Stanton numbers and reattachment Reynolds numbers were calculated, with the distance x_p being given directly in the computed results. Figure 2 shows a comparison between the experimental correlation line obtained by Bushnell and Weinstein and the results of the present calculations. They are seen to be in excellent agreement. The determination of the initial conditions for the interaction calculations demands that $[M_\infty{}^3(c)^{1/2}/(Re_x)^{1/2}]$ is a small quantity where M_∞ is the freestream Mach number, c is the Chapman constant, and Re_x is the Reynolds number based on the distance from the leading edge. This fact necessitates the use of high freestream Reynolds numbers so that the computed results tend to be grouped to the right of Fig. 2.

The encouraging results of Fig. 2 prompted a more detailed study of the theoretical correlation parameters in the hope that the correlation technique itself might be improved. This exercise produced the useful though not totally unexpected result that for a fixed value of s_w the reattachment Reynolds number Re_{x_p} was only a function of the freestream Reynolds number Re_∞. These relationships are

$$s_w = -0.4 \qquad Re_{x_p} = 0.27(Re_\infty)^{0.82}$$

$$s_w = -0.8 \qquad Re_{x_p} = 0.013 Re_\infty$$

They are plotted in Fig. 3.

If the generality of these equations could be increased to cover various freestream conditions and flat plate/wedge geometries, then in practical cases the reattachment Reynolds number could be calculated directly, obviating the need to estimate x_p.

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Applications of Bolotin's Method to Vibrations of Plates

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VARIOUS techniques have been brought to bear on the problem of determining natural frequencies and modes of free transverse vibration of thin elastic plates. Generally, these methods (finite difference, finite element, Rayleigh-Ritz, Fourier series) do not provide accurate numerical results for other than the lowest modes without a substantial, and perhaps prohibitive, investment in digital computation. On the other hand an asymptotic technique developed by Bolotin[1–4] over a decade ago for eigenvalue problems in rectangular regions has the property of greater accuracy for higher modes than for lower modes. Not only does Bolotin's method complement the usual approximate methods of analysis for linear eigenvalue problems, but Bolotin and his co-workers found it to yield, with remarkable accuracy, fundamental natural frequencies for isotropic, and in one instance orthotropic, rectangular plates with supported edges. The purposes of this Note are to call attention to Bolotin's method which has not been widely discussed in the Western literature, to compare results obtained from it with recently published and apparently accurate natural frequencies of orthotropic plates, and to discuss its applicability to problems in which a plate is not supported on some part of its boundary.

We restrict our attention to the case of a uniform rectangular plate lying in the xy-plane $(0 < x < a, 0 < y < b)$ such that the axes of symmetry of the orthotropic material are parallel to the edges of the plate. It will be clear from the analysis that follows that these restrictions, along with the requirement that boundary conditions be uniform along any edge, are necessary to the applicability of the asymptotic method. For free vibration the transverse displacement $w(x, y)\sin\omega t$ is governed by[5]

$$D_x(\partial^4 w/\partial x^4) + 2H(\partial^4 w/\partial x^2\,\partial y^2) + D_y(\partial^4 w/\partial y^4) - \rho\omega^2 w = 0$$

where D_x, H, D_y are flexural rigidities and ρ is the mass per unit area of the plate.

The essence of Bolotin's technique is the assumption that an eigenfunction can be represented adequately by $w = \sin k_x(x - x_o)\sin k_y(y - y_o)$ except near the edges, where the form of an eigenfunction is determined by boundary conditions. This "interior" solution yields a formula for the natural frequency, ω, in terms of the wave numbers k_x and k_y

$$\omega^2 = (1/\rho)[k_x{}^4 D_x + 2k_x{}^2 k_y{}^2 H + k_y{}^4 D_y]$$

We now seek solutions to the governing equation to describe an eigenfunction near edges of the plate. To represent an eigenfunction near $x = 0$ or $x = a$, we let

$$w(x, y) = \phi(x)\sin k_y(y - y_o)$$

hence ϕ is governed by

$$\phi'''' - (2Hk_y{}^2/D_x)\phi'' - k_x{}^2[k_x{}^2 + (2Hk_y{}^2/D_x)]\phi = 0$$

and with $\phi = Ae^{\gamma x}$

$$\gamma^2 = -k_x{}^2$$

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or

$$\gamma^2 = \beta^2 \equiv k_x^2 + (2Hk_y^2/D_x)$$

Requiring that this "edge" solution differ significantly from the interior solution only near an edge, we take

$$\phi = Ae^{-\beta x} + \sin k_x(x - x_o)$$

to represent the eigenfunction near $x = 0$ and

$$\phi = Be^{\beta(x-a)} + \sin k_x(x - x_o)$$

to represent the eigenfunction near $x = a$. Imposition of two homogeneous boundary conditions at each of $x = 0$ and $x = a$ results in a transcendental characteristic equation relating the wave numbers k_x and k_y. For example, if the plate is clamped ($w = 0, \partial w/\partial x = 0$) at $x = 0$ and at $x = a$, then the characteristic equation is

$$\tan k_x a = (D_x/H)(k_x/k_y)[(k_x^2/k_y^2) + (2H/D_x)]^{1/2}$$

The same procedure is followed to produce solutions valid near $y = 0$ and $y = b$. For example if the plate is clamped at $y = b$ and simply supported ($w = 0, \partial^2 w/\partial x^2 = 0$) at $y = 0$ the resulting characteristic relation is

$$\cot k_y b = \left[1 + \left(\frac{2H}{D_y}\right)\left(\frac{k_x^2}{k_y^2}\right)\right]^{1/2}$$

Solution of these equations for a wave number pair (k_x, k_y) results in an estimate of natural frequency and mode of free vibration.

It might be noted at this point that Bolotin's asymptotic method requires the same computational effort for the determination of each of the natural frequencies of the plate and that numerical calculations are feasible with a desk calculator. Furthermore, the larger the wave numbers, k_x and k_y, the larger the β, and hence the narrower the edge zone in which the natural mode significantly deviates from the interior solution. This is the basis for the expectation that the higher modes will be estimated with greater accuracy than will be the lower modes.

Numerical results for the first five natural frequencies for each of two cases of a square orthotropic plate are presented in Table 1 along with values reported by Dickinson.[5] For the case in which all four edges of the plate are clamped, agreement is excellent even for the fundamental frequency. For the case in which the plate is clamped along one pair of opposite edges and free along the other pair, agreement is again excellent except that two frequencies are missed by the asymptotic method. The missing frequencies correspond to beam-like modes, the frequencies of which may be estimated by treating the one-dimensional problem of cylindrical bending.

Fundamental frequencies of square orthotropic plates with supported edges and various bending rigidities are given in Table 2. The results of the asymptotic analysis are in remarkable agreement with those of Dickinson[5] and of Kanazawa and Kawai[6] in spite of the fact that the asymptotic method should give its poorest performance in the computation of fundamental frequencies. Moreover, the results do not suggest any conflict with

Table 1 Natural frequencies of an orthotropic square plate ($D_x/H = 1.543$, $D_y/H = 4.810$, $D_{xy}/H = 0.4070$)

Boundary conditions	Asymptotic analysis $k_x a$	$k_y a$	$\omega a^2(\rho/H)^{1/2}$	Ref. 5 $\omega a^2(\rho/H)^{1/2}$
All edges clamped	1.361	1.451	57.86	58.98
	2.446	1.396	95.99	96.92
	1.244	2.484	141.6	142.0
	3.474	1.296	164.8	165.4
	2.366	2.448	167.5	168.3
Clamped				27.73
($x = 0, x = a$)	1.450	0.772	32.79	31.63
	1.342	1.591	66.19	64.56
Free				76.40
($y = 0, y = a$)	2.477	0.864	82.50	81.77

Table 2 Fundamental frequencies of orthotropic square plates with supported edges

Boundary conditions	D_x/H	D_y/H	Asymptotic analysis $k_x a$	$k_y a$	$\omega a^2(\rho/H)^{1/2}$	Ref. [ ] $\omega a^2(\rho/H)^{1/2}$
All edges clamped	0.5	0.5	1.268	1.268	27.47	28.07 [5]
	0.5	1.0	1.254	1.346	31.56	32.27 [5]
	0.5	2.0	1.244	1.410	38.54	39.29 [5]
	1.0	1.0	1.333	1.333	35.09	35.98 [5]
	1.0	2.0	1.323	1.400	41.42	42.40 [5]
	2.0	2.0	1.392	1.392	46.83	47.96 [5]
Clamped	0.5	0.5	1.134	1.134	21.98	22.49 [6]
($x = 0, y = 0$)	0.5	1.0	1.131	1.170	24.55	25.19 [6]
	0.5	2.0	1.128	1.200	28.99	29.79 [6]
Simply supported	1.0	1.0	1.167	1.167	26.87	27.10 [6]
($x = a, y = a$)	1.0	2.0	1.164	1.198	30.97	31.91 [6]
	2.0	2.0	1.196	1.196	34.58	35.68 [6]

Bolotin's contention[2] that the asymptotic method yields lower bounds for natural frequencies.

The apparent failure of the asymptotic method to predict beam-like modes of vibration is worthy of further discussion because, in addition to the fundamental mode, the estimate of which might be expected to be inaccurate, higher modes are missed. Moreover, the writers have been unable to find evidence that Bolotin and his co-workers have investigated cases other than those for which transverse displacements vanish everywhere on the boundary of the plate. For the sake of simplicity and because of the wealth of numerical results of other investigations, we restrict our attention here to isotropic plates.

Consider the case in which the edges $x = 0$ and $x = a$ are free. For modes symmetric about $x = a/2$ the interior solution may be written

$$w = \cos k_x(x - a/2) \sin k_y(y - y_o)$$

Near the edge $x = 0$

$$w = [Ae^{-\beta x} + \cos k_x(x - a/2)] \sin k_y(y - y_o)$$

where

$$\beta = (k_x^2 + 2k_y^2)^{1/2}$$

Requiring that the bending moment at $x = 0$ vanish,

$$A = [(k_x^2 + \nu k_y^2)/(k_x^2 + (2 - \nu)k_y^2)] \cos(k_x a/2)$$

where ν = Poisson's ratio.

Additionally requiring that the Kirchhoff shear at $x = 0$ vanish, there results the characteristic equation

$$\tan(k_x a/2) = -[(k_x^2 + \nu k_y^2)/(k_x^2 + (2 - \nu)k_y^2)]^2 (1 + 2k_y^2/k_x^2)^{1/2}$$

For any wave number k_y, the smallest root, k_x, of the above is $\pi/a < k_x < 2\pi/a$, for which $A < 0$. Hence the first symmetric mode (for a given k_y) predicted by the asymptotic method is one for which there are two nodal lines approximately paralleling the y-axis. Consequently, all of the beam-like modes are missed. Generally, one expects the adequacy of the asymptotic method to be determined by the magnitude of the decay constant β, i.e. the larger the β, the narrower the edge zone or "boundary layer." The preceding analysis shows that the beam-like modes are missed regardless of k_y (and hence β).

In order to gain some appreciation for the accuracy that might be expected from the asymptotic method when some portion of the boundary of a plate is not supported, natural frequencies were calculated for several cases of square plates with two or three free edges (Table 3). The gaps in Table 3 correspond to the beam-like modes which occur in the first two cases. Results of the asymptotic analysis are in excellent agreement with those compiled by Leissa[7] except for the lowest modes, and even there the accuracy is probably adequate for many engineering applications. The case in which two adjacent edges are clamped is of particular interest because of the commensurate modes predicted by the asymptotic method. Suitable linear combinations of these modes yield displacement patterns either symmetric or antisymmetric with respect to the diagonal $x = y$, and, of course, the actual mode shapes of the plate can all be so classified because

Table 3 Natural frequencies of square isotropic plates with at least two free edges ($\nu = 0.3$)

Boundary conditions	Asymptotic analysis $k_x a$	$k_y a$	$\omega a^2(\rho/D)^{1/2}$	Ref. 7 $\omega a^2(\rho/D)^{1/2}$
Clamped				22.17
($x = 0, x = a$)	1.417	0.838	26.73	26.40
	1.280	1.696	44.56	43.6
Free				61.2
($y = 0, y = a$)	2.465	0.860	67.29	67.2
	1.190	2.598	80.60	79.8
	2.381	1.807	88.17	87.5
Clamped				3.430(L), 3.473(U)[a]
($x = 0$)	0.544	0.702	7.788	7.260(L), 8.547(U)
				20.87(L), 21.30(U)
Free	0.498	1.553	26.27	26.50(L), 27.29(U)
($x = a, y = 0, y = a$)	1.528	0.844	30.06	28.55(L), 31.17(U)
	1.546	1.733	53.21	51.50(L), 54.26(U)
				60.25(L), 61.28(U)
	0.449	2.518	64.54	64.2
	2.513	0.860	69.63	71.1
	1.535	2.641	92.08	
	2.537	1.815	96.06	92.14(L), 97.21(U)
Clamped	0.545	0.545	5.866	6.958(U)
($x = 0, y = 0$)	0.501	1.512	25.05	24.80(U)[b]
	1.512	0.501	25.05	26.80(U)[c]
Free	1.545	1.545	47.13	48.05(U)
($x = a, y = a$)	0.449	2.504	63.87	63.14(U)[b]
	2.504	0.449	63.87	

[a] (L) and (U) denote lower and upper bounds.
[b] Mode antisymmetric with respect to $x = y$.
[c] Mode symmetric with respect to $x = y$.

of the structural symmetry with respect to this diagonal. Young's results[8] reproduced by Leissa[7] and given in Table 3, while not revealing commensurate modes, include two natural frequencies (second and third) which differ by only eight %. The asymptotic analysis similarly suggests that the fifth and sixth natural frequencies are almost equal, but unfortunately the writers have been unable to find values reported for frequencies beyond the fifth.

Although it appears that the lower frequencies are not so accurately predicted as when the transverse displacement vanishes everywhere on the plate boundary, and bearing in mind that beam-like modes are overlooked, we conclude that Bolotin's method is a useful technique for the estimation of natural frequencies of a plate when a portion of the boundary is unsupported.

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Calculation of Compressible Turbulent Free Shear Layers

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Introduction

NOTWITHSTANDING its well-known shortcomings, Prandtl's mixing length theory has been a valuable engineering tool for the prediction of mean velocity fields in turbulent shear layers. Recently Rudy and Bushnell,[1] employing different values of normalized mixing length for planar and axisymmetric flows, satisfactorily predicted a wide range of free turbulent flows except with large sustained density differences. Considering the success of mixing length theory in compressible wall shear flows,[2,3] Rudy and Bushnell suggested the neglect in their solution procedure of turbulence induced transverse static pressure gradients as a possible cause of the poor predictions in supersonic free shear layers with large density differences. Data reported by Brown and Roshko[4] for binary gas mixing show very small effects of density variation on the spreading of a turbulent mixing layer in low-speed flow. They postulated that the large differences in spreading observed in supersonic free mixing layers is an effect of Mach number rather than an effect of density variation. The motivation of this Note is to resolve some of the questions raised previously and to extend the applicability of Rudy and Bushnell's mixing length theory to include supersonic free turbulent mixing.

In the following analysis the normal momentum equation is coupled with the conventional equations of motion and solved iteratively. Results show the transverse static pressure variation has very little direct effect on the mean flow variables in compressible free turbulent shear layers. Indirect effects of static pressure variation through correlations of the fluctuating pressure and fluctuating velocity field (also mentioned by Rudy and Bushnell as a possible mechanism responsible for the observed decreased mixing at higher Mach number) have to be assessed by higher order closure methods not considered herein. However, recently Bradshaw[5] obtained improved predictions of skin friction for supersonic turbulent boundary layers with pressure gradient by including a "mean dilatation effect" in the shear stress equation. His success prompted the present effort to apply a "mean dilatation" correction factor to the mixing length to see if the Mach number effect on spreading rate of a free shear layer would be correctly accounted for. Results show that the use of a mixing length corrected by a properly weighted average value of Bradshaw's[5] mean dilatation factor predicts supersonic turbulent free shear layers correctly.

Analysis

We consider the turbulent homogeneous mixing of two-dimensional or axisymmetric supersonic jets with a low-speed gas as shown schematically in the insert on Fig. 1. With the usual approximation that streamwise derivatives are small compared with normal derivatives, the conservation equations for the mean quantities may be written in the following form:

$$\partial \bar{\rho}\tilde{u}/\partial x + y^{-j}\,\partial y^j \bar{\rho}\tilde{v}/\partial y = 0 \tag{1}$$

$$\bar{\rho}\tilde{u}\,\partial\tilde{u}/\partial x + \bar{\rho}\tilde{v}\,\partial\tilde{u}/\partial y + y^{-j}\,\partial y^j(\overline{\rho u'v'} - R_e^{-1}\bar{\mu}\,\partial\tilde{u}/\partial y)/\partial y = 0 \tag{2}$$

$$\partial\bar{p}/\partial y + \partial\overline{\rho v'^2}/\partial y = 0 \tag{3}$$

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